POLYNOMIAL LIE ALGEBRAS $sl_{pd}(2)$ IN ACTION: SMOOTH $sl(2)$ MAPPINGS AND APPROXIMATIONS

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Abstract

We examine applications of polynomial Lie algebras $sl_{pd}(2)$ to solve physical tasks in G_{inv}-invariant models of coupled subsystems in quantum physics. A general operator formalism is given to solve spectral problems using expansions of generalized coherent states, eigenfunctions and other physically important quantities by power series in the $sl_{pd}(2)$ coset generators $V_{\pm}$. We also discuss some mappings and approximations related to the familiar $sl(2)$ algebra formalism. On this way a new closed analytical expression is found for energy spectra which coincides with exact solutions in certain cases and, in general, manifests an availability of incommensurable eigenfrequencies related to a nearly chaotic dynamics of systems under study.

1 Introduction. General remarks

Recently, in a series of our papers /1-4/ a new efficient Lie-algebraic approach has been suggested to solve both spectral and evolution problems for some nonlinear G_{inv}-invariant models of coupled subsystems in quantum physics. It was based on exploiting a formalism of polynomial Lie algebras g_{pd} as dynamic symmetry algebras g^{DS} of models under study, and besides generators of these algebras g_{pd} can be interpreted as G_{inv}-invariant "essential collective dynamic variables" in whose terms model dynamics are described completely. Specifically, this approach enabled to develop some efficient techniques for solving physical tasks in the case $g^{DS} = sl_{pd}(2)$ when model Hamiltonians H are expressed as follows

$$H = aV_0 + gV_+ + g^*V_- + C = \mathbf{AV} + C, \quad [V_\alpha, C] = 0, \quad V_- = (V_+)^+, \tag{1.1}$$

where C is a function of G_{inv}-dependent model integrals of motion R_i and $V_0, V_{\pm}$ are the $sl_{pd}(2)$ generators satisfying the commutation relations

$$[V_0, V_{\pm}] = \pm V_{\pm}, \quad [V_-, V_+] = \psi_n(V_0 + 1) - \psi_n(V_0), \tag{1.2}$$

with the structure function $\psi_n(V_0)$ being a polynomial $\psi_n(V_0) = A(\{R_j\}) \prod_{i=1}^{n}(V_0 - \lambda_i(\{R_j\}))$ of the degree n in V_0.

For example, the three-boson model Hamiltonian

$$H_2 = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \omega_3 a_3^+ a_3 + g(a_1^+ a_2^+) a_3 + g^*(a_1 a_2) a_3^+ \tag{1.3}$$

can be expressed in the form (1.1) if using the substitutions

$$V_0 = (N_1 + N_2 - N_3)/3, \; V_+ = (a_1^+ a_2^+) a_3, \quad a = \omega_1 + \omega_2 - \omega_3, \quad N_i = a_i^+ a_i,$$

$$2C = R_1(\omega_1 - \omega_2) + R_2(\omega_1 + \omega_2 + 2\omega_3),\ R_1 = N_1 - N_2,\ 3R_2 = N_1 + N_2 + 2N_3 \qquad (1.4a)$$

In this case the structure function $\psi_n(V_0) \equiv \psi_3(V_0)$ is given as follows

$$\psi_3(V_0) = \frac{1}{4}(2V_0 + R_2 - R_1)(2V_0 + R_1 + R_2)(-V_0 + R_2 + 1) \qquad (1.4b)$$

All techniques developed were based on using expansions of physically important quantities (evolution operators, generalized coherent states (GCS), eigenfunctions etc.) by power series in the $sl_{pd}(2)$ coset generators $V_\pm$. Besides, in the Schroedinger picture one has exploited decompositions

$$L(H) = \sum_{\oplus} L([l_i]), \quad (V_+V_- - \psi_n(V_0))|_{L([l_i])} = 0 \qquad (1.5)$$

of Hilbert spaces $L(H)$ of quantum states of model in direct sums of the subspaces $L([l_i])$ which are irreducible with respect to joint actions of algebras $sl_{pd}(2)$ and groups G_{inv} and describe specific "$sl_{pd}(2)$-domains" evolving independently in time under action of the Hamiltonians (1.1). The subspaces $L([l_i])$ are spanned by basis vectors

$$|[l_i]; v\rangle = [(\psi_n(l_0 + v))^{(v)}]^{-1/2} V_+^v |[l_i]\rangle, \quad (\psi_n(x))^{(v)} \equiv \prod_{r=0}^{v-1} \psi_n(x - r),$$

$$V_0|[l_i]; v\rangle = (l_0 + v)|[l_i]; v\rangle,\ R_i|[l_i]; v\rangle = l_i|[l_i]; v\rangle,\ \psi_n(R_0) \equiv \psi_n(V_0) - V_+V_- \qquad (1.6)$$

where $|[l_i]\rangle$ is the lowest vector $(V_-|[l_i]\rangle = 0, \psi_n(l_0) = 0)$ of $L([l_i])$.

Then, using Eqs. (1.1)-(1.2) one may get Jacobi-type three-term recurrence relations for amplitudes $Q_v(E_f) \equiv \langle [l_i]; v|E_f\rangle$ of expansions of energy eigenstates $|E_f\rangle$ in bases $\{|[l_i]; v\rangle\}$. Besides, energy spectra $\{E_f\}$ of bound states are given by roots of certain spectral functions (polynomials for the compact version $sl_{pd}(2) = su_{pd}(2)$) which are determined for given structure functions $\psi_n(x)$ with the help of similar recurrence relations /1-3/. Another way, exploiting the Bargmann-type representation of the $sl_{pd}(2)$ generators,

$$V_+ = z,\ V_0 = zd/dz + l_0,\ V_- = z^{-1}\psi_n(zd/dz + l_0), \qquad (1.7a)$$

reduces these tasks to solving some singular differential equations /1-3/. When using a conjugate to (1.7a) representation of the $sl_{pd}(2)$ generators,

$$V_- = d/dz,\ V_0 = zd/dz + l_0,\ V_+ = \psi_n(zd/dz + l_0)(d/dz)^{-1} \qquad (1.7b)$$

this way leads to solving the Riccati-type equations for structure functions $\psi_3(x)$ of the degree $n = 3$ (that is the case for the Hamiltonian (1.3)). In the paper /4/ some integral expressions were found for amplitudes $Q_v(E)$, eigenenergies $\{E_a\}$ and evolution operators $U_H(t)$ with the help of a specific "dressing" (mapping) of solutions of some auxiliary exactly solvable tasks with the dynamic algebra $sl(2)$.

However, all these and other results do not yield simple working formulas for analysis of models (1.1) and revealing different physical effects (e.g., collapses and revivals of the Rabi oscillations /5/) at arbitrary initial quantum states of models. Besides, solutions /4/ of spectral tasks manifest so-called "quantum discontinuities" /6/: a disappearence of wave

functions when attaining the limit of auxiliary $sl(2)$ Hamiltonians that makes difficult to compare completely quantum models with their semi-classical analogs. Therefore, it is necessary to develop some simple techniques enabling to display important physical peculiriaties of models (1.1)-(1.2). In the case $g^{DS} = sl(2)$, when the structure functions $\psi_n(x) \equiv \psi_2(x)$ are quadratic functions $\psi_2(x) = (j \pm x)(\mp j + 1 - x),\ l_0 = \mp j)$, the GCS formalism of the group orbit type /7/ is known to be an efficient tool for analyzing both linear /7/ and non-linear /7-9/ models. This formalism based on properties of the $SL(2)$ group displacement operators $S_V(\xi) = \exp(\xi V_+ - \xi^* V_-),\ \xi = r\exp(-i\theta)$ yields exact solutions /7/ for linear models and variational schemes (corresponding to the Ehrenfest theorem) to obtain effective mean-field approximate solutions for non-linear models /8-9/.

Below we examine some possibilities of generalizations of this formalism for solving spectral problems of models (1.1)-(1.2) (Section 2) and give a variational scheme to find "smooth" $sl(2)$-approximations of these solutions (Section 3) using an isomorphism of the $sl_{pd}(2)$ algebras to special subalgebras of the extended enveloping algebra $\mathcal{U}(sl(2))$ of the familiar algebra $sl(2)$. This isomorphism is established via a generalized Holstein-Primakoff mapping given as follows /1-3/

$$Y_0 = V_0 - l_0 \mp j,\ j = \frac{s}{2},\ Y_+ = V_+\sqrt{\frac{(j \mp Y_0)(\pm j + 1 + Y_0)}{\psi_n(V_0 + 1)}},\ Y_- = (Y_+)^+, \tag{1.8}$$

where Y_α are the $sl(2)$ generators, upper and lower signs correspond to the $su(2)$ and $su(1,1)$ algebras respectively. In Section 4 some prospects of further studies along these lines are briefly outlined.

2 A general operator formalism to solve spectral problems

As is known /7/, the Hamiltonians (1.1) are simply diagonalized with the help of operators

$$S_V(\xi) = \exp(\xi V_+ - \xi^* V_-) = \exp[t(r)e^{i\theta}V_+]\exp[-2\ln c(r)V_0]\exp[-t(r)e^{-i\theta}V_-],\quad \xi = re^{i\theta} \tag{2.1}$$

when V_α are generators of the familiar $sl(2)$ algebra ($t(r) = \tan r, c(r) = \cos r$ for $su(2)$ and $t(r) = \tanh r, c(r) = \cosh r$ for $su(1,1)$). Indeed, using the well-known $sl(2)$ transformation properties of operators V_α one finds the transformation

$$H \longrightarrow \tilde{H}(\xi) = S_V(\xi)HS_V(\xi)^\dagger = C + V_0A_0(a,g;\xi) + V_+A_+(a,g;\xi) + V_-A_+^*(a,g;\xi) \tag{2.2a}$$

of the Hamiltonians (1.1) under the action of operators $S_V(\xi)$. Then, supposing $A_+(a,g;\xi) = 0$ we find a value ξ_0 of the parameter ξ diagonalizing the Hamiltonian $\tilde{H}(\xi)$. For example, in the case of the $su(2)$ algebra we have /7/

$$\tilde{H}(\xi_0) = S(\xi_0)HS(\xi_0)^\dagger = C + V_0\sqrt{a^2 + 4|g|^2},\quad \xi_0 = \frac{g}{2|g|}arctg\frac{2|g|}{a}, \tag{2.2}$$

and the corresponding eigenenergies $E([l_i];v;\xi_0)$ and eigenfunctions $|[l_i];v;\xi_0\rangle$ are expressed as follows

$$a)\ E([l_i];v;\xi_0) = C + (-j + v)\sqrt{a^2 + 4|g|^2}, \tag{2.3a}$$

$$b)\ |[l_i];v;\xi_0\rangle = S_V(\xi_0)^\dagger |[l_i];v;\rangle = \exp(-\xi_0 V_+ + \xi_0^* V_-)|[l_i];v;\rangle = (\cos^2 r)^{j-v}\times$$

$$\sum_{f\geq 0} \frac{(-e^{i\theta} tgr)^{f-v}}{(f-v)!} F(-v,-v+2j+1;f-v+1;\sin^2 r)[\frac{(2j-v)!f!}{(2j-f)!v!}]^{1/2}|[l_i];f;\rangle \qquad (2.3b)$$

where $F(...)$ is the Gauss hypergeometric function. An equivalent way /8/ to obtain the results (2.3) is based on using the stationarity conditions

$$\frac{\partial E([l_i];v;\xi)}{\partial \theta} = 0, \quad \frac{\partial E([l_i];v;\xi)}{\partial r} = 0 \qquad (2.4)$$

for the energy functional $E([l_i];v;\xi) = \langle [l_i];v;\xi|H|[l_i];v;\xi\rangle$ defined with the help of the $SL(2)$ GCS $|[l_i];v;\xi\rangle = S_V(\xi)^\dagger |[l_i];v;\rangle$ as trial functions.

Both ways above essentially exploit the finite-dimensionality of the $sl(2)$ adjoint (vector) representation (cf. Eq. (2.1a)) and well-known (due to Eq. (2.1)) explicit expansions of the $SL(2)$ GCS in orthonormalized basis states. However, for polynomial Lie algebras $sl_{pd}(2)$ the situation is more complicated since their adjoint representations defined by repeated commutations of arbitrary elements are infinite-dimensional as it follows from Eq. (1.2). Furhtermore, GCS exponential operators $S_V(\xi) = \exp(\xi V_+ - \xi^* V_-)$ have not explicit expressions for matrix elements in orthonormalized bases (1.6) as these exponentials are not elements of Lie groups but only correspond to quasigroups (pseudogroups) /10/ which have no simple analogs of the "disentangling theorem" (2.1) providing expansions of operators $S_V(\xi)$ in finite products of one-parameter subgroups /10, 11/. Therefore, in this case a direct generalization of results (2.3) is impossible.

Nevertheless, taking into account Eqs. (1.2), (1.8) one may apply the diagonalizing scheme (2.2) using represntations of diagonalizing operators $S(\xi)$ by power series

$$S(\xi) = \sum_{f=-\infty}^{\infty} V_+^f S_f(V_0;\xi), \quad V_+^{-k} \equiv V_-^k([\psi_n(V_0)]^{(k)})^{-1},\ k>0 \qquad (2.5)$$

with undetermined (unlike those for the $sl(2)$ algebra - cf. (2.1) and (2.3)) coefficients $S_f(V_0;\xi)$ (which, when being known, provide possibilities of explicit calculations of any physical quantities with the help of Eqs. (1.2), (1.6)). For diagonalizing operators $S(\xi) = S_V(\xi) = \exp(\xi V_+ - \xi^* V_-)$ (if they exist) these coefficients may be taken in the form $S_f(V_0;\xi) = \exp(if\theta)\sigma_f(V_0;r),\ \xi = r\exp(i\theta)$, and satisfy the equations

$$\frac{\partial \sigma_f(V_0;r)}{\partial r} - \sigma_{f-1}(V_0;r) + \psi_n(V_0+f)\sigma_{f+1}(V_0;r) = \delta(r)\delta_{f,0}, \quad f = 0,1,... \qquad (2.6)$$

whose solutions may be represented by power series in r (via direct expansions of exponents $S_V(\xi)$) or obtained in an integral form with the help of the "$sl(2)$ dressing" procedure /4/. In general cases these coefficients satisfy the equations

$$\sum_{f=-\infty}^{\infty} [\psi(V_0)]^{(f)} S_{k+f}(V_0-f;\xi) S_f^*(V_0-f;\xi) = \delta_{k,0} \qquad (2.7)$$

following from the unitarity conditions $SS^\dagger = S^\dagger S = I$.

Then, substituting Eq. (2.5) in the scheme (2.2) one gets after some algebra nonlinear analogs of Eqs. (2.2)

$$a)\,\tilde{H}(\xi) = S_V(\xi) H S_V(\xi)^\dagger = C + \sum_{f=-\infty}^{\infty} V_+^f \tilde{h}_f(V_0;\xi), \quad V_+^{-k} \equiv V_-^k([\psi(V_0)]^{(k)})^{-1},\ k>0,$$

$$\tilde{h}_f(V_0;\xi) = \sum_{k=-\infty}^{\infty} [\psi(V_0)]^{(k-f)} S_k(V_0+f-k;\xi)[a(V_0+f-k)S^*_{k-f}(V_0+f-k;\xi)+$$

$$g\psi(V_0+f-k)S^*_{k+1-f}(V_0+f-k-1;\xi) + g^* S^*_{k-1-f}(V_0+f-k+1;\xi)],$$

$$\tilde{h}_{-f}(V_0;\xi) = \tilde{h}^*_f(V_0-f;\xi)[\psi(V_0)]^{(f)}, \quad [\psi(V_0)]^{(-f)} \equiv ([\psi(V_0+f)]^{(f)})^{-1}, \quad f>0, \qquad (2.8a)$$

$$b)\,\tilde{H}(\xi_0) = S(\xi_0) H S(\xi_0)^\dagger = C + \tilde{h}_0(V_0;\xi_0),\ E([l_i];v;\xi) = C + \langle [l_i];v|\tilde{h}_0(V_0;\xi)|[l_i];v\rangle \quad (2.8b)$$

expressed in terms of the coefficients $S_f(V_0;\xi)$ (hereafter the subscript n in $\psi_n(V_0)$ will be omitted for the sake of the notation simplicity). As is seen from Eq. (2.8b) the diagonalized Hamiltonian $\tilde{H}(\xi_0)$ has (unlike (2.2b)) an essentially non-linear dependence in V_0 determined by coefficients $S_f(V_0;\xi_0)$ which satisfy (additionally to Eqs. (2.7)) the operator recurrence relations following from the condition $S(\xi_0)H = \tilde{H}(\xi_0)S(\xi_0)$),

$$S_f(V_0;\xi_0)[aV_0 - \tilde{h}_0(V_0+f;\xi_0)] + gS_{f-1}(V_0+1;\xi_0) + g^* S_{f+1}(V_0-1;\xi_0) = 0,\ f = 0,\pm1,\pm2,..,$$

$$\tilde{h}_0(V_0;\xi_0) = aV_0 + \sum_{n=-\infty}^{\infty} [\psi(V_0)]^{(n)} S_n(V_0-n;\xi_0)[-nS^*_n(V_0-n;\xi_0)+$$

$$g\psi(V_0-n)S^*_{n+1}(V_0-n-1;\xi_0) + g^* S^*_{n-1}(V_0-n+1;\xi_0)] \qquad (2.9a)$$

or the operator equations

$$0 = \sum_{n=-\infty}^{\infty} [\psi(V_0)]^{(n-f)} S_n(V_0+f-n;\xi_0)[a(V_0+f-n)S^*_{n-f}(V_0+f-n;\xi_0)+$$

$$g\psi(V_0+f-n)S^*_{n+1-f}(V_0+f-n-1;\xi_0) + g^* S^*_{n-1-f}(V_0+f-n+1;\xi_0)],\ f = \pm1,\pm2,.. \quad (2.9b)$$

resulting from the condition $\tilde{h}_f(V_0;\xi_0) = 0,\ f = \pm1,\pm2,..$ (a direct generalization of the condition $A_+(a,g;\xi) = 0$ in (2.2)). Note that Eqs. (2.9), in general, determine both a suitable functional form of $S_f(V_0;\xi)$ and a value ξ_0 of the parameter ξ diagonalizing the Hamiltonian $\tilde{H}(\xi)$.

So, the formalism of the $sl_{pd}(2)$ algebras enabled to to get a general operator scheme of diagonalizing the Hamiltonians (1.1) with the help of solving the (infinite) set of algebraic operator equations (2.7)-(2.9). Evidently, without using some specifications of diagonalizing operators $S(\xi)$ the task of solving these equations is equivalent to that for finding amplitudes $Q_v(E_f) = \langle [l_i];v|S^\dagger|[l_i];f\rangle = [(\psi_n(l_0+f))^{(f)}/(\psi_n(l_0+v))^{(v)}]^{1/2} S^*_{f-v}(l_0+v;\xi_0)$ as Eqs. (2.9b) resemble those for $Q_v(E_f)$. Note that in the case of the compact $su_{pd}(2)$ algebra, having finite-dimensional (with dimensions equal to $d([l_i])$) irreducible subspaces $L([l_i])$, it is possible to simplify the task restricting oneself by the consideration of Eqs. (2.7)-(2.9) on each $L([l_i])$ **independently**. Then, due to the relation $(V_\pm)^{d([l_i])+1}|_{L([l_i])} = 0$ all series in Eqs. (2.5), (2.7)-(2.9) are terminating. Specifically, wave eigenfunctions $|E_f\rangle = S^\dagger|[l_i];f\rangle$ may

be represented by polynomials $|E_f\rangle = A(V_0)\prod_j(V_+ - \Lambda_j(V_0))$ and the energy functionals $E([l_i]; f)_S \equiv \langle [l_i]; f|SHS^{\dagger}|[l_i]; f\rangle$ are written down in the form of a sum of $d([l_i])$ spectral functions as it is prescribed for such classes of models by the algebraic Bethe ansatz /12/. (In essence, we obtain in such a manner a new formulation of this ansatz for a wide class of models in terms of the $su_{pd}(2)$ algebras which is simpler and more efficient (cf. /3/) in comparison with its initial version /12/.)

However, even such simplifications enable us to get simple closed expressions only for little dimensions $d([l_i])$ of the $su_{pd}(2)$ irreducible subspaces $L([l_i])$. At the same time many physical quantum states of models (1.1), e.g., such as coherent and squeezed states in models (1.3), have non-zero projections on all subspaces $L([l_i])$. Therefore, for physical applications it is necessary to get some closed expressions like Eqs. (2.3) for energy eigenvalues and wave eigenfunctions which would describe main features of model dynamics with a good accuracy. One exampl of such analytical approximations was obtained in /1-3/ by mapping (with the help of the change $V_\alpha \rightarrow Y_\alpha$) Hamiltonians (1.1) in Hamiltonians $H_{sl(2)}$ which are linear in $sl(2)$ generators Y_α and have on each fixed subspace $L([l_i])$ equidistant energy spectra given by formulas like Eq. (2.3a) (but with modified constants a, g). However, this (quasi)equidistant approximation is suitable for little or very big dimensions $d([l_i])$ and does not enable to display many peculiarities (e.g., availability and a fine structure of collapses and revivals of the Rabi oscillations) of models (1.1). Therefore, below we describe an alternative approximation applying the variational scheme (2.4) with $SL(2)$ GCS as trial functions to Hamiltonians (1.1) expressed with the help of Eqs. (1.8) as functions of $sl(2)$ generators Y_α.

3 A variational scheme of determining energy spectra with the help of $SL(2)$-coherent states

Hamiltonians (1.1) re-written in terms of Y_α have the form

$$H = aY_0 + gY_+\sqrt{\frac{\psi_n(V_0+1)}{(j \mp Y_0)(\pm j + 1 + Y_0)}} + g^*\sqrt{\frac{\psi_n(V_0+1)}{(j \mp Y_0)(\pm j + 1 + Y_0)}}Y_- + C + a(\pm j + l_0) \tag{1.1'}$$

which is essentially non-linear in $sl(2)$ generators Y_α. Therefore, in general, it is unlikely to diagonalize them with the help of operators $S_Y(\xi) = \exp(\xi Y_+ - \xi^* Y_-)$. However, it is natural to apply associated with these operators $SL(2)$ GCS

$$|[l_i]; v; \xi\rangle = S_Y(\xi)^{\dagger}|[l_i]; v;\rangle = \exp(-\xi Y_+ + \xi^* Y_-)|[l_i]; v;\rangle, \tag{3.1}$$

as trial functions in the variational scheme (2.4) that results in non-linear analogs of Eq. (2.3a) for approximate energy eigenvalues. Such an approximation may be called as a "smooth" $sl(2)$ approximation since it, in fact, corresponds to picking out a "smooth" (due to analytical nature of $SL(2)$ group elements) $sl(2)$ factor $\exp(\xi Y_+ - \xi^* Y_-)$ in the exact diagonalizing operators $S(\tilde{\xi})$.

Specifically, application of this procedure to Hamiltonians with the $su_{pd}(2)$ dynamic symmetry yields after some algebra the following expressions

$$E([l_i]; v; \xi_0) = C + a(l_0 + j) + a(-j + v)\cos 2r - 2|g|(\cos^2 r)^{2(j-v)}\frac{(2j-v)!}{v!}\sum_{f\geq 0} E_v^{\psi}(l_0, j; f),$$

$$E_v^{\psi}(l_0,j;f)=\frac{(tgr)^{2(f-v)+1}(f+1)!}{(f-v)!(f+1-v)!(2j-f-1)!}\sqrt{\frac{\psi(l_0+1+f)}{(2j-f)(f+1)}}\times$$

$$F(-v,-v+2j+1;f-v+1;\sin^2 r)F(-v,-v+2j+1;f-v+2;\sin^2 r) \qquad (3.2)$$

for energy eigenvalues $E([l_i];v;\xi_0=re^{i\theta})$ where $e^{i\theta}=g/|g|$ due to the second condition (2.4) and diagonalizing values of the parameter r are determined from solving the equations

$$0=\sum_{f\geq 0}\frac{\alpha^{2f}}{(2j-1-f)!f!}\{\frac{a\alpha}{|g|}-[4\alpha^2 j-(1+\alpha^2)(2f+1)]\sqrt{\frac{\psi(l_0+1+f)}{(2j-f)(f+1)}}\},\ \alpha=-tgr \quad (3.3)$$

resulting from the first condition (2.4).

As is seen from Eq. (3.2), spectral functions $E_v^{\psi}(l_0,j;f)$ are non-linear in the discrete variable v labeling energy levels that provides a non-equdistant character of energy spectra within fixed subspaces $L([l_i])$ at $d([l_i])>3$; besides, due to the availability of square roots in expressions for these functions different eigenfrequencies $\omega_v\equiv E_v/\hbar$ are incommensurable: $m\omega_{v_1}\neq n\omega_{v_2}$ that is an indicator of an origin of collapses and revivals of the Rabi oscillations /5/ as well as of prechaotic dynamics. We also note that Eqs. (3.2)-(3.3) give exact results at little dimensions $d([l_i])$.

4 Conclusion

So, we have obtained a general operator scheme for diagonalizing Hamiltonians (1.1) and a smooth approximation for solutions of its defining equations with the help of the mapping (1.8) and the variational scheme (2.4) using the $SL(2)$ GCS as trial functions. This approximation may be used as an initial one in iterative schemes of solving Eqs. (2.9) (re-written in "the $sl(2)$ languauge") which are similiar to those developed to examine non-linear problems of classical mechanics and optics /11/. Further investigations along this line may be also related to a search of suitable specifications of the operators $S(\xi)$ (besides the form $S(\xi)=S_V(\xi)$) reducing solutions of Eqs. (2.9) to determining a value ξ_0 providing an exact or a sufficiently accurate approximation for diagonalization of the Hamiltonian (1.1) in scheme (2.3a). Another way to develop the results above concerns some simplifications of the formulas (3.2) via using different properties, including integral representations, of the hypergeometric functions $F(a,b;c;x)$. For the case of $\psi_3(x)$ it is also of interest to compare results of such approximations with exact calculations obtained by considering exactly solvable cases of the Riccati equations yielded by the $sl_{pd}(2)$ representation (1.7b).

Finally, general ideas of the analysis above may be extended to solve evolution problems. Specifically, a version of general operator formalism for these tasks was formulated in /4/, and a version of obtaining a variational dynamics in the mean-field approximation can be found following the approach of the paper /8/. Namely, looking for the evolution operator $U_H(t)$ (with H given by Eq. (1.1')) in the form $U_H(t)=\exp(-z(t)Y_+ + z(t)^*Y_-)$ and using the $sl(2)$ GCS $|z(t)\rangle=\exp(-z(t)Y_+ + z(t)^*Y_-)|\psi_0\rangle$ as trial functions in the time-dependent Hartree-Fock varitional scheme with the Lagrangian $\mathcal{L}=\langle z(t)|(i\partial/\partial t-H)|z(t)\rangle$ one gets the $sl(2)$ Euler-Lagrange equations

$$\dot{q}=\frac{\partial\mathcal{H}}{\partial p}, \qquad \dot{p}=-\frac{\partial\mathcal{H}}{\partial q} \qquad (4.1)$$

for "motion" of the $sl(2)$ GCS parameters; here $\mathcal{H} = \langle z(t)|H|z(t)\rangle$ and $p = j\cos\theta, q = \phi, z = \tan(\theta/2)\exp(-i\phi)$ for $su(2)$ and $p = j\cosh\theta, q = \phi, z = \tanh(\theta/2)\exp(-i\phi)$ for $su(2)$.

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